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Numerical Grid Generation in Arbitrary Surfaces through a Second-Order Differential-Geometric Model

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In this paper a set of second-order partial differential equations for the generation of coordinates in a given surface has been developed and then solved numerically to demonstrate its potential for surface coordinate generation. The proposed equations are not some arbitrarily chosen equations but are a consequence of the formulae of Gauss and thus carry with them an explicit dependence on the geometric properties of the given surface. Furthermore, these equations are easy to solve and require only the specification of the bounding curves to provide the Dirichlet boundary conditions for their solution. Results of coordinate generation both in simply and doubly connected regions on some known surfaces, with the option of coordinate redistribution, have been presented. Extension of this technique to arbitrary surfaces seems to be straightforward. © 1986 Academic Press, Inc.

I. INTRODUCTION

The problem of generating spatial coordinates by numerical methods through carefully selected mathematical models is of current interest both in mechanics and physics. A review of various methods of coordinate generation in both two- and three-dimensional Euclidean (R^2 and R^3) spaces is available in [1], and reference may also be made to the proceedings of two recent conferences, [2, 3], and a book [4] on the topic of numerical grid generation.

This paper is exclusively directed to the problem of generation of a desired coordinate system in the surface of a given body and thus, in a basic sense, it is an effort directed at the problem of grid generation in a 2D non-Euclidean space. The mathematical model selected for this purpose is based on the formulae of Gauss for a surface and has been discussed by the author in earlier publications, [5–8]. The resulting equations are three coupled second-order quasilinear elliptic partial differential equations with the Cartesian coordinates as the dependent variables. These equations are nonhomogeneous with their right-hand sides depending both on the components of the normal and the mean curvature of the surface; thus reflecting some geometrical aspect of the surface in an explicit manner.

This paper also addresses two very important problems which are germane to the

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proposed mathematical model for surface coordinate generation. The first problem is in regard to the basic structure of the arbitrary specified control functions. This aspect has been analyzed as fully as possible in the Appendix A. The second problem is to establish clearly the connection, if any, between the present model and a model formed by inverting the Laplace system in R^3. This analysis, reported in Appendix B, shows that the two models exactly coincide when the transverse coordinate is orthogonal to the surface. Despite the nonavailability of such mathematical justifications, the proposed equations have earlier been used to generate 3D coordinates between two given surfaces in [9–11].

Previous work on the subject of surface grid generation has been done by using either the algebraic techniques, [12–14], or using the PDE approach, [15–18]. All the algebraic methods depend very heavily on the use of highly accurate interpolating schemes. In the PDE methods, the model in [15] is derived from the 3D Laplace system, which has been discussed more fully in Appendix B of this paper. Some of the results in [16–18] are common and in large part depend on the generation of a coordinate system in a surface based on an already available system.

Numerical solution of the proposed equations depends on the availability of the surface equation in the Cartesian form either as $F(x, y, z) = 0$ or $z = f(x, y)$, and on the prescription of the data on the bounding curves in the surface which eventually form the Dirichlet boundary conditions for the equations.

Numerical solutions of the proposed equations for the coordinates in either simply or doubly connected regions of some known surfaces have been obtained and shown in Figs. 1–4. It has also been shown that any desired control on the distribution of grid spacing can be exercised by a proper choice of the control functions; cf. Fig. 4. Extension of the proposed method to arbitrary surfaces is purely formal.

II. Nomenclature

$b_{\alpha\beta} = \mathbf{n}^{(\nu)} \cdot \mathbf{r}_{,\alpha\beta}$; coefficients of the second fundamental form in the surface $\nu = \text{const}$.
D = second-order differential operator, (Eq. (3.3a)).
$g = \det(g_{ij})$.
$G_\nu = g_{\alpha\alpha} g_{\beta\beta} - (g_{\alpha\beta})^2$, (ν, α, β) cyclic.
g_{ij} = covariant metric components.
g^{ij} = contravariant metric components.
J = Jacobian determinant.
$k_{\mathrm{I}}^{(\nu)}, k_{\mathrm{II}}^{(\nu)}$ = principal curvatures at a point in the surface $\nu = \text{const}$.
$\mathscr{L}$ = second-order differential operator, (Eq. (3.10a)).
$\mathbf{n}^{(\nu)}$ = unit normal vector on the surface $\nu = \text{const}$.
$\bar{P}, \bar{Q}$ = control functions.
$P^{\alpha}_{\beta\gamma}$ = control functions.
$\mathbf{r} = (x, y, z)$.
R defined in (3.3b).

x^i = 3D curvilinear coordinates.
x^α = 2D curvilinear coordinates.
x, y, z = rectangular Cartesian coordinates.
$X_i^{(v)}$ = rectangular components of $\mathbf{n}^{(v)}$; $X_1^{(v)} = X^{(v)}$, $X_2^{(v)} = Y^{(v)}$, $X_3^{(v)} = Z^{(v)}$.
$\Upsilon_{\alpha\beta}^{\delta} = \frac{1}{2} g^{\sigma\delta}(\partial g_{\alpha\sigma}/\partial x^\beta + \partial g_{\beta\sigma}/\partial x^\alpha - \partial g_{\alpha\beta}/\partial x^\sigma)$,
the surface Christoffel symbols of the second kind.
$\Gamma_{ij}^{k} = \frac{1}{2} g^{mk}(\partial g_{im}/\partial x^j + \partial g_{jm}/\partial x^i - \partial g_{ij}/\partial x^m)$,
the space Christoffel symbols of the second kind.
$\Delta_2^{(v)} x^\alpha = -g^{\beta\gamma} \Upsilon_{\beta\gamma}^{\alpha}$,
Beltrami's second-order differential parameter.

(i) *Notation for partial derivatives.*

$$\mathbf{r}_{,\alpha} = \frac{\partial \mathbf{r}}{\partial x^\alpha}, \qquad \mathbf{r}_{,\alpha\beta} = \frac{\partial^2 \mathbf{r}}{\partial x^\alpha \, \partial x^\beta}.$$

Also

$$\mathbf{r}_\xi = \frac{\partial \mathbf{r}}{\partial \xi}, \qquad \mathbf{r}_{\xi\eta} = \frac{\partial^2 \mathbf{r}}{\partial \xi \, \partial \eta},$$

etc.

(ii) *Note on the use of indices.* The Latin indices i, j, k, etc., are used when the index rage is from 1 to 3. The Greek indices α, β, γ, etc. (except v, see below), are used for the cases when the indices assume only two integer values.

$v = 1$: Greek indices α, β, etc., assume integer values 2 and 3.
$v = 2$: Greek indices α, β, assume integer values 3 and 1.
$v = 3$: Greek indices α, β, etc., assume integer values 1 and 2.

(iii) *Summation convention.* In this paper the summation convention on repeated indices is implied when the same index appears both as a lower and as an upper index. Thus the summation convention is implied in T_α^α but not in $T_{\alpha\alpha}$. The summation convention is also suspended when one repeated index is enclosed in the parentheses, e.g., as in $T_\alpha^{(\alpha)}$.

III. The Mathematical Model

The mathematical basis of the present formulation along with the derivation of the model equations which have been used in this paper are available in Refs. [6–8]. However, for the sake of clarity of exposition we list here only the core steps which lead to the final form of the equations. In the ensuing development we shall continuously use the conventions and symbols as stated in Section II of this paper.

The formulae of Gauss (cf. [6, 21]) for a surface $x^{\nu} = \text{const.}$ are compactly written as

$$\mathbf{r}_{,\alpha\beta} = \Upsilon^{\delta}_{\alpha\beta} \mathbf{r}_{,\delta} + \mathbf{n}^{(\nu)} b_{\alpha\beta}. \tag{3.1}$$

Inner multiplication of Eq. (3.1) by $G_{\nu} g^{\alpha\beta}$ results in the availability of a vector differential equation

$$D\mathbf{r} + G_{\nu}(\Delta_2^{(\nu)} x^{\delta})\, \mathbf{r}_{,\delta} = \mathbf{n}^{(\nu)} R, \tag{3.2}$$

where

$$D = G_{\nu}\, g^{\alpha\beta} \partial_{\alpha\beta}, \tag{3.3a}$$

$$R = G_{\nu}\, g^{\alpha\beta} b_{\alpha\beta} = (k_{\mathrm{I}}^{(\nu)} + k_{\mathrm{II}}^{(\nu)})\, G_{\nu}, \tag{3.3b}$$

$$\Delta_2^{(\nu)} x^{\delta} = -g^{\alpha\beta} \Upsilon^{\delta}_{\alpha\beta}. \tag{3.3c}$$

The vector equation (3.2) provides three scalar second-order partial differential equations for the determination of the Cartesian coordinates x, y, z. For a plane ($R = 0$), the Eq. (3.2) reduces to the TTM equations (cf. [1] and the references contained therein), and in this case the Beltramians $\Delta_2 x^{\delta}$ become the Laplacians $\nabla^2 x^{\delta}$.

The quantity $k_1^{(\nu)} + k_2^{(\nu)} = R/G_{\nu}$ is twice the mean curvature at a point of the surface $x^{\nu} = \text{const.}$ and is invariant to a coordinate transformation in the surface. It can be expressed in one of the following two ways: (i) in terms of the partial derivatives of $\mathbf{r}$ with respect to x^{δ} ($\delta \neq \nu$), and (ii) in terms of the partial derivatives of $\mathbf{r}$ with respect to x^{i}, which also includes the transverse coordinate x^{ν}. In the first case

$$k_{\mathrm{I}}^{(\nu)} + k_{\mathrm{II}}^{(\nu)} = g^{\alpha\beta} b_{\alpha\beta}, \tag{3.4a}$$

while for the second case

$$k_{\mathrm{I}}^{(\nu)} + k_{\mathrm{II}}^{(\nu)} = g^{\alpha\beta} \Gamma^{\nu}_{\alpha\beta} \lambda^{(\nu)}, \tag{3.4b}$$

where $\Gamma^{\nu}_{\alpha\beta}$ are the Christoffel symbols of the second kind in the embedding space, with x^{ν} as the transverse coordinate, and

$$\lambda^{(\nu)} = \mathbf{n}^{(\nu)} \cdot \mathbf{r}_{,\nu}. \tag{3.4c}$$

In the second case only first partial derivatives with respect to x^{ν} appear which are assumed to have been evaluated at the surface. There is, however, no restriction on the orthogonality or non orthogonality of x^{ν} to the surface.

The surface-coordinate generating system of equations, with the option of arbitrary coordinate control, is now obtained by putting suitable restriction on the Beltramians appearing in Eq. (3.2). The most general form one can have is to take

$$\Delta_2^{(\nu)} x^{\delta} = g^{\alpha\beta} P^{\delta}_{\alpha\beta}, \tag{3.5}$$

where $P^{\delta}_{\alpha\beta}$ are symmetric in the lower two indices and thus represent six *arbitrarily* chosen control functions.

We now impose the following important requirement on the functions $P^{\delta}_{\alpha\beta}$: *If the coordinates are such that the Beltramians vanish, i.e., $\Delta_2^{(v)} x^{\delta} = 0$, then the control functions $P^{\delta}_{\alpha\beta}$ vanish individually for all values of δ, α, β pertaining to that surface.* The importance of this restiction can be felt by equating the right-hand sides of Eqs. (3.3c) and (3.5),

$$g^{\alpha\beta} P^{\delta}_{\alpha\beta} = -g^{\alpha\beta} \Upsilon^{\delta}_{\alpha\beta}, \tag{3.6}$$

where $\Upsilon^{\delta}_{\alpha\beta}$ are the surface Christoffel symbols of the second kind. Thus, when the Beltramians vanish the right-hand side of (3.6) vanishes in the *sense of an inner sum* but the left-hand side vanishes because of the imposed restriction. (It must be noted that in a surface or even in the case of curvilinear coordinates in a plane, all the Christoffels are not zero). Thus $P^{\delta}_{\alpha\beta} \neq \Upsilon^{\delta}_{\alpha\beta}$. For a relation between the values of $P^{\delta}_{\alpha\beta}$ under successive coordinate transformations and also a relation between $P^{\delta}_{\alpha\beta}$ and $\Upsilon^{\delta}_{\alpha\beta}$ refer to Eqs. (A-6), (A-8), and (A-9).

The generating system is now obtained by substituting (3.5) in Eq. (3.2) as

$$D\mathbf{r} + G_v (g^{\alpha\beta} P^{\delta}_{\alpha\beta}) \, \mathbf{r}_{,\delta} = \mathbf{n}^{(v)} R. \tag{3.7}$$

To be specific, we take the surface $x^3 = \zeta = \text{const.}$ as the given surface and $x^1 = \xi$, $x^2 = \eta$. Then using the formulae from the surface theory, $g^{11} = g_{22}/G_3$, $g^{12} = -g_{12}/G_3$, $g^{22} = g_{11}/G_3$, $G_3 = g_{11} g_{22} - (g_{12})^2$, we have

$$\mathscr{L}\mathbf{r} = \mathbf{n}^{(3)} R. \tag{3.8}$$

From Eq. (3.8) the three scalar equations are

$$\mathscr{L}x = X^{(3)} R, \qquad \mathscr{L}y = Y^{(3)} R, \qquad \mathscr{L}z = Z^{(3)} R, \tag{3.9a, b, c}$$

$$\mathscr{L} = g_{22} \partial_{\xi\xi} - 2g_{12} \partial_{\xi\eta} + g_{11} \partial_{\eta\eta} + \bar{P} \partial_{\xi} + \bar{Q} \partial_{\eta}, \tag{3.10a}$$

$$\bar{P} = g_{22} P^1_{11} - 2g_{12} P^1_{12} + g_{11} P^1_{22}, \tag{3.10b}$$

$$\bar{Q} = g_{22} P^2_{11} - 2g_{12} P^2_{12} + g_{11} P^2_{22}. \tag{3.10c}$$

Equations (3.9) are the basic generating equations for the curvilinear coordinates in a given surface. A question which naturally arises at this stage is as follows: What is the connection, if any, between the proposed equations (3.8) for a surface and the set of equations obtained by inverting a set of three Poisson's equations, when one of the three coordinates is kept fixed which defines the same surface? To answer this question we have shown in Appendix B that if the coordinates ξ, η, ζ satisfy the Poisson's equations and the ζ-coordinate is orthogonal to the surface $\zeta =$ const., then the resulting equations are in fact the same as (3.8). This analysis also establishes a connection between the Laplacians and Beltramians at the surface. Equations (3.9a, b) were also obtained through the Poisson's system in Ref. [15] by assuming that the surface be representable as $z = f(x, y)$ and further, besides being orthogonal the ζ-coordinates should also be the lines of zero curvature. The analysis in Appendix B shows that neither the surface equation in the form $z = f(x, y)$ nor the vanishing of curvature of the transverse lines is a prerequiste for obtaining Eqs. (3.8).

IV. Numerical Implementation

Numerical solution of Eqs. (3.9) can be obtained by any suitable numerical method of solution which has proved useful in any elliptic grid generation problem. In this paper the equations have been discritized by using central differences for both the first and second derivatives and then solved iteratively from an initial guess by using the LSOR. The main difference between the coordinates in a flat space and in a surface is the appearance of the right-hand side terms in which the quantity R can be established a priori. This requires a knowledge of the equation of the surface in either the form $F(x, y, z) = 0$ or $z = f(x, y)$. For arbitrary surfaces the equation in the form $F(x, y, z) = 0$ or $z = f(x, y)$ can be established by the least squares method [19], and for further accuracy, more x, y, z values in the surface can be obtained by using the bicubic spline interpolation of Ref. [20]. Some of the known surfaces have been duplicated by using the above techniques while verifying the results of this paper.

Having obtained a twice continuously differentiable form $F(x, y, z) = 0$, the sum of the principal curvatures

$$k_{\mathrm{I}} + k_{\mathrm{II}} = R/G_3$$

can be obtained from the well-known result, e.g., [21],

$$\begin{aligned} k_{\mathrm{I}} + k_{\mathrm{II}} = [&(F_y^2 + F_z^2)(2F_xF_zF_{xz} - F_z^2F_{xx} - F_x^2F_{zz}) \\ &+ 2F_xF_y(F_z^2F_{xy} + F_xF_yF_{zz} - F_yF_zF_{xz} - F_xF_zF_{yz}) \\ &+ (F_x^2 + F_z^2)(2F_yF_zF_{yz} - F_z^2F_{yy} - F_y^2F_{zz})]/P^3F_z^2, \end{aligned} \tag{4.1}$$

where

$$P^2 = F_x^2 + F_y^2 + F_z^2.$$

Formula (4.1) is valid for $F_z \neq 0$. At a point or on a line if $F_z = 0$, then another form of (4.1) can be obtained by replacing x by y, y by z, and z by x in which F_z does not appear in the denominator.

To demonstrate the use of Eqs. (3.9) for the generation of surface coordinates, we have selected three well-known surfaces for the purpose of introducing a desired system of coordinates in them.

(a). *Coordinates in an Elliptic Cylinder Forming a Simply Connected Domain.* This problem is a prototype of a coordinate generation in a given piece of a surface. The region under consideration forms a simply connected region bounded by the space arcs $\eta=\eta_0$, $\eta=\eta_1$, $\xi=\xi_0$, and $\xi=\xi_1$. Here $\eta=\eta_0, \eta_1$ are the elliptical arcs in the xy-plane, and $\xi=\xi_0, \xi_1$, are straight-lines parallel to the z-axis. The equations are:

$$
\begin{aligned}
&\eta=\eta_0: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad z=z_0,\\
&\eta=\eta_1: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad z=z_1,\\
&\xi=\xi_0: x=-a, \qquad y=0,\, z_1 \leqslant z \leqslant z_0,\\
&\xi=\xi_1: x=a, \qquad y=0,\, z_1 \leqslant z \leqslant z_0.
\end{aligned} \tag{4.2}
$$

The Dirichlet boundary conditions are provided by the data of (4.2) for the solution of Eqs. (3.9).

(b) *Coordinates in an Arbitrary Ellipsiod Cut By the Planes* $z=z_0$, $z=z_1$. This case is of coordinate generation in a doubly connected region bounded by two closed space curves on an ellipsoid. The space curves $\eta=\eta_0$ and $\eta=\eta_1$, are given by

$$
\begin{aligned}
&\eta=\eta_0: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad z=z_0,\\
&\eta=\eta_1: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad z=z_1.
\end{aligned} \tag{4.3}
$$

We now imagine a cut joining the curves $\eta=\eta_0$, $\eta=\eta_1$ while still remaining in the surface. As in the 2D case no boundary conditions can be prescribed on the cut line. However, since the values of x, y, z above and below the cut should be the same, we impose the periodicity conditions:

$$
x(\xi_1,\eta)=x(\xi_0,\eta), \qquad y(\xi_1,\eta)=y(\xi_0,\eta), \qquad z(\xi_1,\eta)=z(\xi_0,\eta). \tag{4.4}
$$

The Dirichlet conditions (4.3) and the periodicity conditions (4.4) yield a unique solution to the set of Eqs. (3.9).

(c) *Coordinates in an Arbitrary Elliptic Parabolid Cut By the Planes* $z=z_0$, $z=z_1$. This is again the case of coordinate generation in a doubly connected region bounded by two closed curves on an elliptic parabolid. The space curves $\eta=\eta_0$ and $\eta=\eta_1$ are given by

$$
\begin{aligned}
&\eta=\eta_0: \frac{x^2}{a^2}+\frac{y^2}{b^2}=z, \qquad z=z_0,\\
&\eta=\eta_1: \frac{x^2}{a^2}+\frac{y^2}{b^2}=z, \qquad z=z_1.
\end{aligned} \tag{4.5}
$$

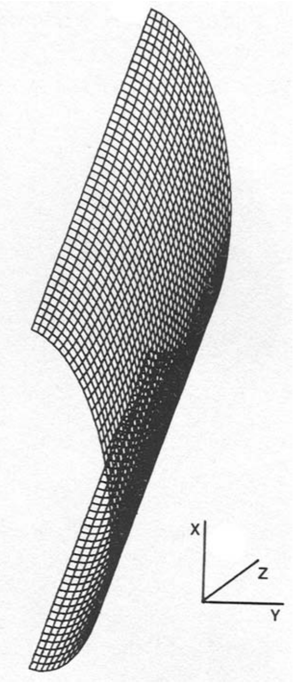

FIG. 1. A simply connected region on the surface of an elliptic cylinder; data; $a = 1.0$, $b = 0.5$, $z_0 = 2.0$, $z_1 = 0.0$.

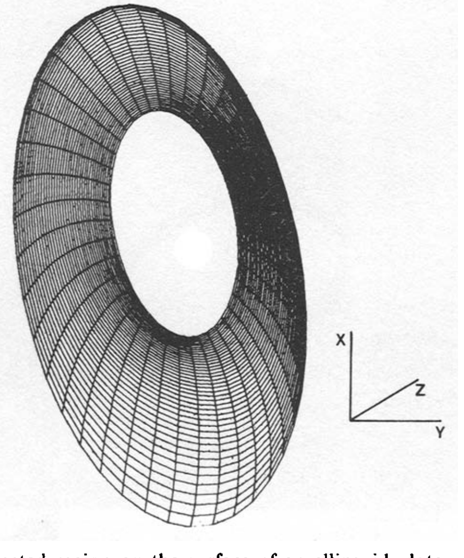

FIG. 2. A doubly connected region on the surface of an ellipsoid; data: $a = 5.0$, $b = 3.0$, $c = 1.0$, and cut by the planes $z_0 = 0.9$, $z_1 = 0.0$.

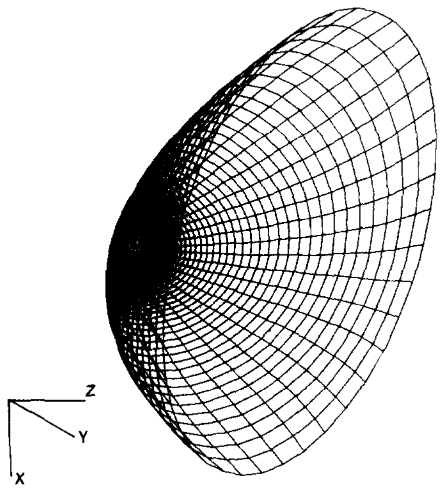

FIG. 3. A doubly connected region on the surface of an elliptic paraboloid; data $a = 2.0$, $b = 1.0$, and cut by the planes $z_0 = 0.01$ and $z_1 = 1.96$.

Under the boundary data (4.5) along with the periodicity conditions (4.4) the Eqs. (3.9) have been solved.

In all cases (a)–(c) the control functions $P^1_{\alpha\beta}$ and $P^2_{\alpha\beta}$ have been set equal to zero, i.e., $\Delta_2^{(3)}\xi = 0$, $\Delta_2^{(3)}\eta = 0$, and the results are demonstrated in Figs. 1–3. Figure 4 shows the result of a coordinate concentration near the curve $z = z_0 = 0.9$ of case (b). In this case we have taken

$$P^1_{\alpha\beta} = 0, \qquad P^2_{11} = P^2_{12} = 0$$

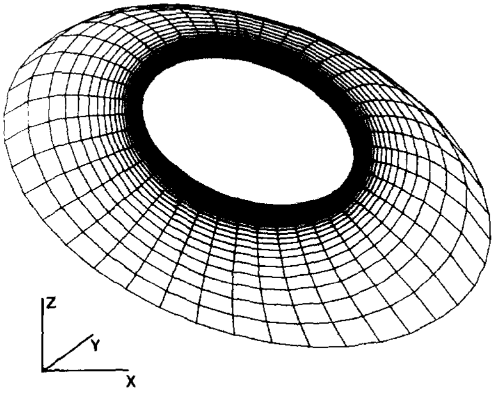

FIG. 4. Data and configuration same as in Fig. 2. Coordinate contraction near $z = z_0$ with $\kappa = 1.1$.

and

$$P^2_{22} = -(2.0 + (\eta - \eta_0)\ln\kappa)\ln\kappa/(1.0 + (\eta - \eta_0)\ln\kappa), \tag{4.6}$$

where $\kappa = 1.1$ is a constant.

The computer programs which have been developed to solve the Eqs. (3.9) have been used successfully for all the case enumerated above both with and without coordinate contraction. Also each case in which the surface can be represented as $z = f(x, y)$ was repeated to determine whether it is necessary to solve the z-equation too along with the x and y-equation. It has been found that solving all the three equations (3.9) or solving only Eqs. (3.9a, b) while taking the z iteratively from $z = f(x, y)$ produces almost the same results.

V. Conclusions

A set of second order partial differential equations has been developed and then solved numerically to generate the coordinates in a given surface. The proposed equations are a logical outcome of the formulae of Gauss and thus explicitly depend on some basic differential-geometric properties of the surface in which the coordinates are to be introduced. The proposed method of surface coordinate generation is simple to implement and is capable of extension to arbitrary surfaces.

Appendix A

In this appendix we further elaborate on the concept of the control functions $P^\delta_{\alpha\beta}$ as introduced in Section III, and study their transformation law starting from the *principle* that each one of them vanishes individually when the coordinates satisfy the equations $\Delta_2 x^\delta = 0$. Further, the relation between $P^\delta_{\alpha\beta}$ and $\Upsilon^\delta_{\alpha\beta}$ has also been established.

As a first step we consider Eq. (3.1), which on inner multiplication by $g^{\alpha\beta}$ yields

$$g^{\alpha\beta}\left(\frac{\partial^2 \mathbf{r}}{\partial x^\alpha\,\partial x^\beta} + P^\delta_{\alpha\beta}\frac{\partial \mathbf{r}}{\partial x^\delta}\right) = \mathbf{n}^{(\nu)}(k^{(\nu)}_{\mathrm{I}} + k^{(\nu)}_{\mathrm{II}}), \tag{A.1}$$

where according to (3.6),

$$g^{\alpha\beta}P^\delta_{\alpha\beta} = \Delta^{(\nu)}_2 x^\delta.$$

Let $x^\alpha_{(m-1)}$ and $x^\alpha_{(m)}$, where $m \geqslant 1$, be two successive coordinate transformations with the transformation Jacobian not equal to zero. Also, we take $x^\alpha_{(0)}$ as those coordinates for which

$$\Delta^{(\nu)}_2 x^\alpha_{(0)} = 0$$

and then the equations

$$g^{\alpha\beta}_{(0)} P^{\delta}_{\alpha\beta(0)} = -g^{\alpha\beta}_{(0)} \Upsilon^{\delta}_{\alpha\beta(0)} = 0 \tag{A.2}$$

are assumed to be satisfied by taking

$$P^{\delta}_{\alpha\beta(0)} = 0 \tag{A.3}$$

for all pertinent values of δ, α, β.

For the coordinates $x^{\alpha}_{(m-1)}$, $x^{\alpha}_{(m)}$, Eq. (A.1) is

$$\begin{aligned} g^{\alpha\beta}_{(m-1)} &\left(\frac{\partial^2 \mathbf{r}}{\partial x^{\alpha}_{(m-1)}\, \partial x^{\beta}_{(m-1)}} + P^{\delta}_{\alpha\beta(m-1)} \frac{\partial \mathbf{r}}{\partial x^{\delta}_{(m-1)}} \right) \\ &= \mathbf{n}^{(\nu)}_{(m-1)} (k^{(\nu)}_{\mathrm{I}} + k^{(\nu)}_{\mathrm{II}})_{(m-1)}, \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} g^{\alpha\beta}_{(m)} &\left(\frac{\partial^2 \mathbf{r}}{\partial x^{\alpha}_{(m)}\, \partial x^{\beta}_{(m)}} + P^{\delta}_{\alpha\beta(m)} \frac{\partial \mathbf{r}}{\partial x^{\delta}_{(m)}} \right) \\ &= \mathbf{n}^{(\nu)}_{(m)} (k^{(\nu)}_{\mathrm{I}} + k^{(\nu)}_{\mathrm{II}})_{(m)}. \end{aligned} \tag{A.5}$$

We now use the transformation law

$$g^{\alpha\beta}_{(m-1)} = \frac{\partial x^{\alpha}_{(m-1)}}{\partial x^{\gamma}_{(m)}} \frac{\partial x^{\beta}_{(m-1)}}{\partial x^{\mu}_{(m)}} g^{\gamma\mu}_{(m)}$$

and the chain rule of differentiation for the first and second partial derivatives of $\mathbf{r}$ in (A.4). With this done and noting that both $\mathbf{n}$ and $k^{(\nu)}_{\mathrm{I}} + k^{(\nu)}_{\mathrm{II}}$ are invariant to coordinate transformation, we compare the equation obtained from (A.4) with Eq. (A.5). This comparison yields

$$P^{\delta}_{\alpha\beta(m)} = \left[P^{\sigma}_{\varepsilon\mu(m-1)} \frac{\partial x^{\delta}_{(m)}}{\partial x^{\sigma}_{(m-1)}} + \frac{\partial^2 x^{\delta}_{(m)}}{\partial x^{\varepsilon}_{(m-1)}\, \partial x^{\mu}_{(m-1)}} \right] \frac{\partial x^{\varepsilon}_{(m-1)}}{\partial x^{\alpha}_{(m)}} \frac{\partial x^{\mu}_{(m-1)}}{\partial x^{\beta}_{(m)}}. \tag{A.6}$$

Another form can be obtained by introducing

$$\frac{\partial x^{\alpha}_{(m-1)}}{\partial x^{\beta}_{(m)}} = C^{\alpha}_{\beta}/J, \tag{A.7}$$

where for each fixed ν the four C^{α}_{β} are obtained from the 3D formula

$$C^{i}_{j} = \frac{\partial x^{r}_{(m)}}{\partial x^{s}_{(m-1)}} \frac{\partial x^{p}_{(m)}}{\partial x^{t}_{(m-1)}} - \frac{\partial x^{r}_{(m)}}{\partial x^{t}_{(m-1)}} \frac{\partial x^{p}_{(m)}}{\partial x^{s}_{(m-1)}},$$

and this (i, s, t), (j, r, p) are to be taken in the cyclic permutations of 1, 2, 3. Further

$$J = \det\left(\frac{\partial x^{\alpha}_{(m)}}{\partial x^{\beta}_{(m-1)}}\right).$$

Then

$$P^{\delta}_{\alpha\beta(m)} = \left[P^{\sigma}_{\varepsilon\mu(m-1)} \frac{\partial x^{\delta}_{(m)}}{\partial x^{\sigma}_{(m-1)}} + \frac{\partial^2 x^{\delta}_{(m)}}{\partial x^{\varepsilon}_{(m-1)}\, \partial x^{\mu}_{(m-1)}}\right] C^{\varepsilon}_{\alpha} C^{\mu}_{\beta}/(J)^2. \tag{A.8}$$

Either one of the equations (A.6), (A.8) is a recursive relation connecting the two sets of values of $P^{\delta}_{\alpha\beta}$ in passing from one coordinate system to another, starting from $P^{\delta}_{\alpha\beta(0)} = 0$. These equations explicitly show the effect of a coordinate transformation on the control functions.

To establish a relation between $P^{\delta}_{\alpha\beta}$ and $\Upsilon^{\delta}_{\alpha\beta}$, we use the formula

$$\frac{\partial^2 x^{\delta}_{(m)}}{\partial x^{\varepsilon}_{(m-1)}\, \partial x^{\mu}_{(m-1)}} = \Upsilon^{\sigma}_{\varepsilon\mu(m-1)} \frac{\partial x^{\delta}_{(m)}}{\partial x^{\sigma}_{(m-1)}} - \Upsilon^{\delta}_{\theta\phi(m)} \frac{\partial x^{\theta}_{(m)}}{\partial x^{\varepsilon}_{(m-1)}} \frac{\partial x^{\phi}_{(m)}}{\partial x^{\mu}_{(m-1)}}$$

in Eq. (A.6) and obtain for $m \geqslant 1$.

$$P^{\delta}_{\alpha\beta(m)} = -\Upsilon^{\delta}_{\alpha\beta(m)} + [P^{\sigma}_{\varepsilon\mu(m-1)} + \Upsilon^{\sigma}_{\varepsilon\mu(m-1)}] \frac{\partial x^{\delta}_{(m)}}{\partial x^{\sigma}_{(m-1)}} \frac{\partial x^{\varepsilon}_{(m-1)}}{\partial x^{\alpha}_{(m)}} \frac{\partial x^{\mu}_{(m-1)}}{\partial x^{\beta}_{(m)}}. \tag{A.9}$$

Inner multiplication of (A.9) by $g^{\alpha\beta}_{(m)}$ proves the validity of Eq. (3.6) for all $m \geqslant 1$.

Appendix B

The purpose of this appendix is to demonstrate that the basic equations (Eqs. (3.2) or (3.9)) can also be obtained from the inversion of the 3D Laplace system in R^3 if the transverse coordinate is assumed to be orthogonal to the chosen surface. From the ensuing analysis it becomes clear that the condition of zero curvature imposed on the transverse coordinate in Ref. [15] is really not needed to obtain the surface equations from the inverted 3D Laplace system. The results obtained here also establish a connection between the 3D Laplacians of coordinate lines in a surface with the corresponding Beltramians.

Let ξ, η, ζ be a general curvilinear coordinate system in R^3 such that ξ, η form the coordinates in the surface $\zeta = \text{const.}$ with ζ as the transverse coordinate. Without imposing the condition of orthogonality of the ζ-coordinate, the inversion of the 3D Laplacians $\nabla^2\xi$, $\nabla^2\eta$, $\nabla^2\zeta$ is given by the equation[1] (refer to Eq. (B.3) of Ref. [1]),

[1] $g^2_{13} = (g_{13})^2$, etc.

$$\begin{aligned}
&g_{22}\mathbf{r}_{\xi\xi} - 2g_{12}\mathbf{r}_{\xi\eta} + g_{11}\mathbf{r}_{\eta\eta} + \mathbf{r}_{\xi}\{-g_{23}^2\Gamma_{11}^1 + 2g_{13}g_{23}\Gamma_{12}^1 - g_{13}^2\Gamma_{22}^1 \\
&\quad + G_3\Gamma_{33}^1 + 2G_5\Gamma_{13}^1 + 2G_6\Gamma_{23}^1 + g\nabla^2\xi\}/g_{33} + \mathbf{r}_{\eta}\{-g_{23}^2\Gamma_{11}^2 \\
&\quad + 2g_{13}g_{23}\Gamma_{12}^2 - g_{13}^2\Gamma_{22}^2 + G_3\Gamma_{33}^2 + 2G_5\Gamma_{13}^2 + 2G_6\Gamma_{23}^2 + g\nabla^2\eta\}/g_{33} \\
&\quad + \mathbf{r}_{\zeta}\{-g_{23}^2\Gamma_{11}^3 + 2g_{13}g_{23}\Gamma_{12}^3 - g_{13}^2\Gamma_{22}^3 + G_3\Gamma_{33}^3 + 2G_5\Gamma_{13}^3 \\
&\quad + 2G_6\Gamma_{23}^3 + g\nabla^2\zeta\}/g_{33} = 0, \qquad \text{(B.1)}
\end{aligned}$$

where

$$g = g_{33}G_3 + g_{13}G_5 + g_{23}G_6,$$

$$G_3 = g_{11}g_{12} - g_{12}^2, \qquad G_5 = g_{12}g_{23} - g_{13}g_{22}, \qquad G_6 = g_{12}g_{13} - g_{23}g_{11},$$

and we have made use of the equations

$$\mathbf{r}_{,ij} = \Gamma_{ij}^p\,\mathbf{r}_{,p}$$

with the Latin indices taking all values from 1 to 3. Also Γ_{ij}^p are the 3-space Christoffel symbols as defined in Section II.

Let $\mathbf{n}^{(3)} = \mathbf{n}$ be the unit normal vector on the surface $\zeta = \text{const}$. Then taking the dot product of Eq. (B.1) with $\mathbf{n}$ (so that $\mathbf{r}_{\xi}\cdot\mathbf{n} = \mathbf{r}_{\eta}\cdot\mathbf{n} = 0$) and writing $\lambda = \mathbf{n}\cdot\mathbf{r}_{\zeta}$, we get

$$\begin{aligned}
g\nabla^2\zeta = &-g_{33}G_3(k_{\text{I}}^{(3)} + k_{\text{II}}^{(3)})/\lambda + g_{23}^2\Gamma_{11}^3 - 2g_{13}g_{23}\Gamma_{12}^3 \\
&+ g_{13}^2\Gamma_{22}^3 - 2G_5\Gamma_{13}^3 - 2G_6\Gamma_{23}^3 - G_3\Gamma_{33}^3. \qquad \text{(B.2)}
\end{aligned}$$

Substituting (B.2) in (B.1) and using the equations

$$\begin{aligned}
g\nabla^2\xi = &-g_{22}g_{33}\Gamma_{11}^1 + g_{23}^2\Gamma_{11}^1 - g_{11}g_{33}\Gamma_{22}^1 + g_{13}^2\Gamma_{22}^1 - 2g_{13}g_{23}\Gamma_{12}^1 \\
&+ 2g_{12}g_{33}\Gamma_{12}^1 - G_3\Gamma_{33}^1 - 2G_5\Gamma_{13}^1 - 2G_6\Gamma_{23}^1, \qquad \text{(B.3)}
\end{aligned}$$

$$\begin{aligned}
g\nabla^2\eta = &-g_{22}g_{33}\Gamma_{11}^2 + g_{23}^2\Gamma_{11}^2 - g_{11}g_{33}\Gamma_{22}^2 + g_{13}^2\Gamma_{22}^2 - 2g_{13}g_{23}\Gamma_{12}^2 \\
&+ 2g_{12}g_{33}\Gamma_{12}^2 - G_3\Gamma_{33}^2 - 2G_5\Gamma_{13}^2 - 2G_6\Gamma_{23}^2, \qquad \text{(B.4)}
\end{aligned}$$

we get

$$\begin{aligned}
&g_{22}\mathbf{r}_{\xi\xi} - 2g_{12}\mathbf{r}_{\xi\eta} + g_{11}\mathbf{r}_{\eta\eta} + (2g_{12}\Gamma_{12}^1 - g_{22}\Gamma_{11}^1 - g_{11}\Gamma_{22}^1)\,\mathbf{r}_{\xi} \\
&\quad + (2g_{12}\Gamma_{12}^2 - g_{22}\Gamma_{11}^2 - g_{11}\Gamma_{22}^2)\,\mathbf{r}_{\eta} = \mathbf{r}_{\zeta}G_3(k_{\text{I}}^{(3)} + k_{\text{II}}^{(3)})/\lambda. \qquad \text{(B.5)}
\end{aligned}$$

Equation (B.5) can also be written as

$$\begin{aligned}
&\mathscr{L}\mathbf{r} + [2g_{12}(\Gamma_{12}^1 - \Upsilon_{12}^1) - g_{22}(\Gamma_{11}^1 - \Upsilon_{11}^1) - g_{11}(\Gamma_{22}^1 - \Upsilon_{22}^1)]\,\mathbf{r}_{\xi} \\
&\quad + [2g_{12}(\Gamma_{12}^2 - \Upsilon_{12}^2) - g_{22}(\Gamma_{11}^2 - \Upsilon_{11}^2) - g_{11}(\Gamma_{22}^2 - \Upsilon_{22}^2)]\,\mathbf{r}_{\eta} \\
&\quad = \mathbf{n}G_3(k_{\text{I}}^{(3)} + k_{\text{II}}^{(3)}) + (\mathbf{r}_{\zeta} - \lambda\mathbf{n})\,G_3(k_{\text{I}}^{(3)} + k_{\text{II}}^{(3)})/\lambda, \qquad \text{(B.6)}
\end{aligned}$$

where

$$\mathscr{L} = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta} + G_3(\Delta_2\xi)\,\partial_\xi + G_3(\Delta_2\eta)\,\partial_\eta$$

as previously defined in (3.10a).

It must again be emphasized that Γ^i_{jk} are the 3-space Christoffel symbols while $\Upsilon^\delta_{\alpha\beta}$ are the 2-space surface Christoffel symbols and in general they are not equal to each other at $\zeta = \text{const}$.

ζ-Coordinates Orthogonal to the Surface. If the ζ-coordinates are orthogonal to the surface $\zeta = \text{const.}$, then

$$g_{13} = g_{23} = 0,$$

and in this case it is easy to show that

$$\Gamma^\alpha_{\beta\gamma} = \Upsilon^\alpha_{\beta\gamma},$$

where all Greek indices assume only values 1 and 2. Also since ζ is normal to the surface,

$$\mathbf{n}^{(3)} = \mathbf{n} = \mathbf{r}_\zeta/\sqrt{g_{33}}, \qquad \lambda = \sqrt{g_{33}}.$$

Thus Eq. (B.6) becomes

$$\mathscr{L}\mathbf{r} = \mathbf{n}G_3(k^{(3)}_{\mathrm{I}} + k^{(3)}_{\mathrm{II}}), \tag{B.7}$$

which is Eq. (3.8); proving the contention set forth at the beginning of this appendix.

Under $g_{13} = g_{23} = 0$, since $g = g_{33}G_3$, we obtain from (B.2)–(B.4) the following results.

$$\begin{aligned} \nabla^2\zeta &= -\Gamma^3_{33}/g_{33} - (k^{(3)}_{\mathrm{I}} + k^{(3)}_{\mathrm{II}})/\sqrt{g_{33}}, \\ \Delta^{(3)}_2\xi &= \nabla^2\xi + \Gamma^1_{33}/g_{33}, \\ \Delta^{(3)}_2\eta &= \nabla^2\eta + \Gamma^2_{33}/g_{33}. \end{aligned} \tag{B.8}$$

The Christoffels Γ^i_{33} are as follows:

$$\Gamma^1_{33} = \left(g_{12}\frac{\partial g_{33}}{\partial\eta} - g_{22}\frac{\partial g_{33}}{\partial\xi}\right)\Big/2G_3,$$

$$\Gamma^2_{33} = \left(g_{12}\frac{\partial g_{33}}{\partial\xi} - g_{11}\frac{\partial g_{33}}{\partial\eta}\right)\Big/2G_3,$$

$$\Gamma^3_{33} = \frac{\partial g_{33}}{\partial\zeta}\Big/2g_{33}.$$

Using (3.4b) it is a straightforward matter to show that

$$k_{\rm I}^{(3)} + k_{\rm II}^{(3)} = -\left(\frac{\partial G_3}{\partial \zeta}\right)\Big/ 2G_3\sqrt{g_{33}}. \tag{B.9}$$

(Note that all the ζ-derivatives appearing in the preceding formulae are those which have been evaluated at the surface.)

Equations (B.8) show clearly the connections between the Laplacians and the Beltramians.

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